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# Some Open Problems in Parameterized Complexity Related to the Work of Jianer Chen

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**Abstract:** This short paper highlights some open problems related to the work of Jianer Chen in the area of parameterized/multivariate algorithmics.

**Key words:** parameterized complexity; multivariate algorithmics; open problems

## 1 Introduction

When a pioneering scientist turns sixty, it is traditional to offer a collection of papers dedicated to this event, revisiting a life's creative work (hopefully, just an initial part!) and attempting to inspire young researchers about the scientific horizons that opus uncovered.

This essay will necessarily fail to be comprehensive in these regards, but surely, some of these open problems exposed in Jianer Chen's researches in this area are destined to become lighthouse problems in the further developments of this successful new angle on algorithms and complexity.

It is important to note that Jianer has engaged an impressively wide range of research areas in computer science, both theoretical and applied.

This short essay about open problems, addresses only just one of his research areas: his contributions and inspiring open problems in multivariate algorithmics.

## 2 $O^*(f(k))$ Races and Their Limits

A parameterized problem is *fixed-parameter tractable* (FPT) if it can be solved in time $f(k)n^c$ for some function $f$ and constant $c$, where $n$ is the size of a problem instance. This is written as $O^*(f(k))$ ignoring the polynomial part of the running time.

Since the beginnings of the field, there has been continual interest in finding "better and better" FPT algorithms for famous concrete problems, meaning by this, FPT algorithms with "improved" $f(k)$. For example, the first FPT algorithm for the FEEDBACK VERTEX SET problem, due to Downey and Fellows[1] (see also Bodlaender[2]) had a running time of $O^*((2k+1)^k)$. The current best result for this problem is an FPT algorithm that runs in time of $O^*(3.83^k)$, due to Cao, Chen, and Liu[3].

Jianer Chen and his co-authors hold an impressive number of the current best-known records in these $O^*(f(k))$ "races", including the best current result on the primordial VERTEX COVER problem, for which the best known FPT algorithm, due to Chen, Kanj, and Xia. runs in time $O^*(1.2738^k)$[4]

This raises the question, first discussed at one of the early Dagstuhl workshops, concerning whether such improvements could go on "forever". Could it possibly happen that there might be a series of ever-better FPT algorithms for VERTEX COVER with running times like $O^*(1.1^k)$, $O^*(1.01^k)$, $O^*(1.001^k), \cdots$, possibly with the polynomials hiding in the $O^*(\cdot)$ notation being of ever-larger degree?

The surprising answer is "No."

**Theorem 1** Assuming the Exponential Time Hypothesis (ETH), there exists a positive constant $\xi_{VC} > 0$, such that no strongly FPT parameterized algorithm for VERTEX COVER can run in time $O^*((1+\xi_{VC})^k)$.

A neat proof due to Yijia Chen can be found in Downey and Fellows[5] as Theorem 29.5.9.

| **Open Problem 1** |
| --- |
| Find any sort of quantitative lower bound information about this mysterious barrier constant $\xi_{VC}$ for the VERTEX COVER problem. We currently know only that it exists (and is bounded above by 0.2738). |

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## 3 Parameterized Analog of Cook's Theorem and XP-Optimality

One of the most important early papers in the foundations of parameterized complexity, by Cai, Chen, Downey, and Fellows[6] investigated "The parameterized complexity of short computation and factorization," and established a parameterized analog of the famous Cook's Theorem of classical complexity.

Cook's Theorem is important in the classical setting because it connects a polynomial time version of the NONDETERMINISTIC TURING MACHINE (NDTM) HALTING PROBLEM (essentially, the defining problem for NP) to 3SAT: a convenient starting point for hardness reductions.

Parameterized complexity began with a circuit complexity definition of W[1], and early-on it was shown that CLIQUE is complete for W[1][1], thus providing a convenient starting point for hardness reductions in the parameterized setting.

A key part of the picture provided by Cai, Chen, Downey, and Fellows[6] is that the following natural parameterized form of the HALTING PROBLEM is also complete for W[1], completing the analog to Cook's Theorem.

**Problem:** SHORT NDTM COMPUTATION

*Instance:* A nondeterministic single-tape Turing Machine $\mathcal{M}$ (with unbounded nondeterminism and unbounded alphabet size), and a positive integer $k$.

*Parameter:* $k$

*Question:* Is it possible for $\mathcal{M}$ to halt in at most $k$ steps?

In about 2005, the study of what is now known as XP-optimality was launched by Juedes, Chor and Fellows[7], quickly followed up by Chen and co-authors[8] with pioneering results:

**Theorem 2** There is no $O(n^{o(k)})$ algorithm for the $k$-CLIQUE problem unless FPT = M[1]. Similarly, there is no $O(n^{(o(k)})$ algorithm for the $k$-DOMINATING SET problem unless FPT = M[2].

The significance of the above theorem is that it shows that the brute force try-all-$k$-subsets algorithm that runs in time $O(n^{O(k)})$ is essentially optimal. Assuming what is known as the Strong Exponential Time Hypothesis (SETH), an even stronger optimality result holds for $k$-DOMINATING SET[9].

**Theorem 3** Assuming SETH, there is no $O(n^{k-\epsilon})$ algorithm for $k$-DOMINATING SET for any $\epsilon > 0$, where $n$ is the number of vertices of the input.

It is known that FPT = M[1] if and only if ETH fails[10].

**Open Problem 2**
Can it be shown that SETH fails if and only if FPT = M[2]?

These two seminal directions of Jianer Chen's research on the foundations of parameterized complexity are now set to collide.

In their two books and various papers, Downey and Fellows have often suggested that SHORT NDTM COMPUTATION was probably stuck at the level of the brute force try-all-length-$k$-computation-paths algorithm that runs in time $O(n^{O(k)})$, because Turing machines are such unstructured and opaque objects.

This is probably not correct! As recently pointed out by Dániel Marx and Saket Saurabh (private communications, 2014), there is a little bit of structure that can be exploited: the planar computational tableaux imposed by the tape of the Turing machine.

**Open Problem 3**
What is the optimal XP complexity of SHORT NDTM COMPUTATION? Can it be solved in time $O(n^{O(\sqrt{k})})$? What about the other problems, such as SHORT POST CORRESPONDENCE, studied in the foundational paper of Cai, Chen, Downey, and Fellows[6]?

## 4 Parameterization and Approximation

In 1997, Liming Cai and Jianer Chen published the first paper exploring connections between parameterized complexity and approximation complexity[11]. This was the main subject of Liming Cai's PhD dissertation supervised by Jianer.

**Theorem 4** If an NP-optimization problem has a fully polynomial time approximation scheme, then it is FPT[11-13].

Interest in the possible interactions of parameterized complexity and approximation complexity remain strong, as these two approaches to computational complexity seem most generally successful. How can they be combined?

An important open problem area that explores beyond the results of Section 3 is termed *FPT Approximation*.

**Open Problem 4**
Is there an FPT algorithm that outputs correctly, either:
(1) $G$ has no $k$-vertex dominating set, or
(2) $G$ has a dominating set of size at most $2^k$.
(Of course, one could ask about a different function of $k$.) An analogous question concerning $k$-INDEPENDENT SET is also open.

Recently, Jianer Chen and Iyad Kanj published a pioneering paper combining XP-optimality and the complexity of approximation[14]. They proved the following theorem:

**Theorem 5** Unless the ETH hypothesis fails, the DISTINGUISHING SUBSTRING SELECTION problem has no Polynomial Time Approximation Scheme (PTAS) with running time $f(1/\epsilon)m^{o(1/\epsilon)}$ for any function $f$[14].

**Open Problem 5**
Prove similar lower bounds for other problems that admit PTASs.

## 5 Guess-and-Check, Lower Bound Machineries, and Limited Nondeterminism

In a pioneering paper in 1993[15], Liming Cai and Jianer Chen explored the connections between models of computation with limited nondeterminism (such as introduced by Kintala and Fischer[16], Buss and Goldsmith[17], and Papadimitriou and Yannakis[18], and parameterized complexity classes, based on their "guess-and-check" model.

Intuitively, the model allows some bits of information to be "guessed" about the input of size $n$, and this is traded off against the power of the verifying mechanism. With reference to parameterized complexity classes, W[1] allows $O(k \log n)$ bits to be guessed, and verified by weft 1 circuits. W[2] allows $O(k \log n)$ bits to be guessed, and verified by weft 2 circuits, etc.

An early but important observation of Liming Cai is that FPT coincides with $f(k)$ guessed bits, verified by a polynomial time algorithm[6]. In essence, this either closely presages, or was the first published proof of a lemma that has assumed central importance in recent years.

**Lemma** A parameterized problem is FPT if and only if it is kernelizable in polynomial time[6, 19, and general folklore].

In the guess-and-check framework, the advice tells you what to do with the kernel.

Powerful lower bound machineries have emerged from the general principle:

> *Too much concentration of "the truth" leads to collapses of plausible complexity hierarchies.*

Two examples:

(1) Michael Dinneen showed in his 1996 dissertation[20] (and published this result in Refs. [21,22]) that for an NP-hard parameterized graph problem where for each fixed parameter value $k$, the YES-instances form a minor order ideal (and thus the set of characterizing obstructions $O_k$ is finite by the Graph Minor Theorem), we cannot have any polynomial-in-$k$ bound on $|O_k|$ unless NP $\subseteq$ coNP/poly.

(2) Burhman and Hitchcock in 2008[23] showed that NP-hard sets are exponentially dense unless NP $\subseteq$ coNP/poly.

The recently developed methods for proving lower bounds for FPT kernelization exploit the above principle[24, 25]. The guess-and-check framework suggests these methods might be generalized further.

The GRAPH BANDWIDTH problem is known to be hard for W[$t$] for all $t$[26], but it is not known to belong to W[P]. Intuition suggests it shouldn't (as first pointed out by Michael Hallett[27]). If it did belong to W[P], then one would need only $k \log n$ guessed bits and a polynomial-time verifier (as W[P] is represented in the guess-and-check model).

But then, if I wanted to know that *all* of the graphs $G_i$, for $i = 1, 2, \cdots, m$, have bandwidth at most $k$, then apparently I only need to guess $k \log N$ bits and polynomially verify, where $N = \Sigma_{i=1}^{m} \log |G_i|$ bits, rather than the expected $k \cdot \Sigma_{i=1}^{m} \log |G_i|$ bits, by asking for the evidence that the graph $G$ formed as the disjoint union of the $G_i$ has bandwidth at most $k$.

**Open Problem 6**
Can we develop lower bound machinery for higher levels of the W-hierarchy? In particular, can we show that W[$t$]-hard compositional problems (such as BANDWIDTH) cannot be in W[P] unless NP $\subseteq$ coNP/poly (or something similar)?

## 6 Conclusions

This small paper has visited only a few of Jianer Chen's seminal contributions to the development of parameterized/multivariate algorithmics. I particularly salute his warm and infectious enthusiasm, and generosity (and apparently millions of frequent flyer miles) establishing the subject in China and collaborating around the world. I recall with warmth our early investigations of the subject, and look forward to many more years of collaboration. Happy birthday, Jianer.

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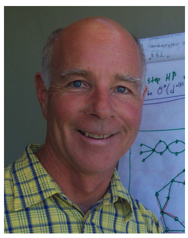

**Michael Fellows** Professor Michael R. Fellows (Charles Darwin Univ., Australia), received his PhD degree in computer science from the Univ. California, San Diego (his hometown) in 1985. He is well-known for his foundational work in parameterized complexity (Springer Festschrift LNCS 7370) and has received numerous awards, including an Alexander von Humboldt Research Prize (in 2006) and the Nerode Prize (in 2014). In 2014, he was honored as one of the ten inaugural Fellows of the EATCS.