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A Piecewise Linear Programming Algorithm for Sparse Signal Reconstruction

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Abstract: In order to recover a signal from its compressive measurements, the compressed sensing theory seeks the sparsest signal that agrees with the measurements, which is actually an l_0 norm minimization problem. In this paper, we equivalently transform the l_0 norm minimization into a concave continuous piecewise linear programming, and propose an optimization algorithm based on a modified interior point method. Numerical experiments demonstrate that our algorithm improves the sufficient number of measurements, relaxes the restrictions of the sensing matrix to some extent, and performs robustly in the noisy scenarios.

Key words: compressed sensing; continuous piecewise linear programming; interior point method

1 Introduction

The foundation of Compressed Sensing (CS) theory is laid on the three papers[1–3] that inspired a burst of intensive research activities over the years. CS provides an alternative to Shannon/Nyquist sampling for the acquisition of sparse or compressible signals that can be well approximated by just k ($\ll n$) components from an n-dimensional basis. In this framework one does not measure the n-dimensional signal directly, but rather inner products with $m\,(\ll n)$ measurement vectors and then recovers the signal via certain reconstruction algorithms. This small ratio of m/n makes it possible to simplify the sensing system. Hence, the implications of these facts are far-reaching, with applications in sensor networks[4, 5], medical imaging[6], data compression[7], analog-to-digital converters[8, 9], single-pixel camera[10], and so on.

The essential issue in the CS theory is the signal reconstruction. Although the recovery of a signal from the extremely limited measurements appears to be a severely ill-posed inverse problem, the prior knowledge of sparsity gives us a solid hope for accurate reconstruction. Actually, the signal recovery can be achieved by searching for the sparsest one that agrees with the observed measurements.

Mathematically speaking, under the sparsity and noise-free assumptions, one can recover a k-sparse signal $\bar{\boldsymbol{x}} \in \mathbb{R}^n$, namely $\|\bar{\boldsymbol{x}}\|_0 \leqslant k$ (e.g., the coefficient sequence of the signal in an appropriate basis), by solving the nonconvex optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \|\boldsymbol{x}\|_0, \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned} \tag{1}$$

where $\|\cdot\|_0$ denotes the l_0 "norm" that counts the number of non-zero elements, and the sensing matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is usually generated by randomly sampling the columns independently from a certain distribution (e.g., the Gaussian distribution).

Unfortunately, problem (1) is known to be NP-hard and is generally impossible to be solved when problem scale goes large, as it usually requires to perform a combinatorial enumeration of all the feasible sparse situations. However, fundamental results in Ref. [2] showed that, a computationally tractable optimization

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problem yields an equivalent solution, which can be found by solving the Basis Pursuit (BP) problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \| \boldsymbol{x} \|_1, \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned} \tag{2}$$

as long as $\boldsymbol{A}$ satisfies the Restricted Isometry Property (RIP). Problem (2) can be viewed as the closest convexification of problem (1) and is much more approachable, which can be easily solved with linear programming techniques. Shortly afterwards, a burst of researches in sparse signal reconstruction have been motivated by BP and RIP. More and more practical and sophisticated algorithms were proposed.

Kim et al.[11] proposed a specialized interior-point method for solving the l_1-regularized least squares problem, which used the preconditioned conjugate gradients algorithm to compute the search direction. Other methods for this problem include the gradient projection method[12], the Bregman iterative algorithms[13–15], and the shrinkage and subspace optimization[16]. Candès et al.[17] described a method called Iterative Reweighted l_1 minimization (IRL1) consisting of solving a sequence of weighted l_1 minimization problems with fewer measurements than l_1 minimization. Wang and Yin[18] presented an Iterative Support Detection (ISD) method, which runs as fast as the best BP algorithms but requires significantly fewer measurements via solving convex truncated BP.

It is shown by Chartrand[19] that a nonconvex variant of BP could produce exact reconstruction with fewer measurements. Specifically, problem (2) is replaced by the l_p minimization,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \| \boldsymbol{x} \|_p^p, \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned} \tag{3}$$

where $\| \boldsymbol{x} \|_p = \left(\sum_i |\boldsymbol{x}_i|^p \right)^{1/p}$, $0 < p < 1$, is the l_p-quasi-norm of $\boldsymbol{x}$. Then one can adopt a simple computational approach, such as the gradient descent with projection, to compute local minimizers of problem (3). This work is extended and refined in the subsequent years. Chartrand et al.[20–22] considered the use of Iteratively Reweighted Least Squares (IRLS) approach for the above nonconvex problem, and the experiment results suggested that $p = 1/2$ seems suitable. Recently, researchers extended problem (3) to the matrix space, called M_p-minimization[23]. Regarding the nonconvex variants of BP, besides L_p norm, log-sum function is also an effective sparsity-encouraging function which behaves very close to L_0 norm. Iterative reweighted methods and theoretical analysis based on log-sum minimization were studied in a number of works[24–26]. Notice that, the nonconvexity of problem means that all of the algorithms considered here are only designed to produce local optima. However, these local algorithms may give global solutions, if initialized by a point sufficiently close to the global optimum[19].

Gilbert et al.[27] showed the way to incorporate greedy iterative strategies into fast sparse approximation algorithms and establish the first rigorous guarantees for greedy methods. Tropp and Gilbert[28] proved theoretically and empirically that Orthogonal Matching Pursuit (OMP) is effective for CS. Soon after, faster algorithms have been proposed, such as Stagewise OMP (StOMP)[29], Regularized OMP (ROMP)[30], Compressive Sampling Matching Pursuit (CoSaMP)[31], Subspace Pursuit (SP)[32], Iterative Hard Thresholding (IHT)[33], Accelerated Iterative Hard Thresholding (AIHT)[34], and so on. The major advantages of this kind of algorithms are their fast speed and their easiness of implementation.

Cormode and Muthukrishnan[35] presented an approach of two sets of group tests with different separation properties that yields the first known polynomial time explicit construction of a non-adaptive transformation matrix and a reconstruction algorithm. Gilbert et al.[36] exhibited the Chaining Pursuit (CP) method which combines sublinear reconstruction time with stable and robust linear dimension reduction of all compressible signals. However, simulations reveal that CP works well only when the signal is extremely sparse. Subsequently, Gilbert et al.[37] presented Heavy Hitters on Steroids (HHS) pursuit. Unlike CP, HHS uses separate matrices for estimation, sifting, and noise reduction.

Ji et al.[38] considered from a Bayesian perspective and utilized the Relevance Vector Machine (RVM) for signal estimation. Seeger and Nickisch[39] extended these ideas to Bayesian experimental design and provided an approximate method based on expectation propagation. Baron et al.[40] described a specific measurement scheme using an low density parity check like (LDPC-like) measurement matrix or a CS-LDPC measurement matrix, and employed belief propagation techniques to accelerate the reconstruction of approximately sparse signals. The other related methods on application of Bayesian framework to sparse inverse problem can be found in Ref. [41] and the references therein.

By comparing the above approaches, we find that the algorithms based on the nonconvex optimization (such as L_p norm and log-sum minimization) have reliably good performance, including fewer measurements, and less required prior knowledge. The reason is that, these algorithms utilize L_p norm or log-sum function as the objective function to approximate the original L_0 norm. Then a question naturally emerges: whether a different approach based on an equivalent model (to L_0 norm) instead of the approximation models might also find the correct solution?

The main purpose of this paper is to propose such a novel alternative. We consider a new perspective and treat problem (1) as a concave Continuous Piecewise Linear Programming (CPLP) problem based on the mathematical essence of l_0 norm. Next, we propose a modified interior point method, and refer to this method as CS-IPM. Although the concavity and piecewise linearity of the proposed model make CS-IPM only return a local optimum, they provide a new framework that parallels the conventional theory and allows us to address a variety of issues that previously have not been addressed.

The remainder of this paper is organized as follows. Section 2 builds the concave piecewise linear model. The proposed algorithm CS-IPM is outlined in Section 3. In Section 4, a series of numerical experiments are conducted for the comparison of CS-IPM and the other algorithms. Section 5 concludes the paper briefly.

2 Piecewise Linear Model for Signal Reconstruction

Taking advantage of the prior knowledge about the sparsity of the original signal, i.e., $k \in \mathbb{R}$, we simplify problem (1) into the following feasibility problem,

$$\begin{aligned} &\text{find} \quad \boldsymbol{x} \\ &\text{s.t.} \quad \boldsymbol{Ax} = \boldsymbol{b}, \\ &\qquad \| \boldsymbol{x} \|_0 \leqslant k \end{aligned} \tag{4}$$

Consider the following problem,

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} \quad \sum_{i=1}^{n-k} |\boldsymbol{x}|_{[i]}, \\ &\text{s.t.} \quad \boldsymbol{Ax} = \boldsymbol{b} \end{aligned} \tag{5}$$

where $|\boldsymbol{x}|$ is the absolute value of $\boldsymbol{x}$, componentwise (i.e., $|\boldsymbol{x}|_i = |x_i|, i = 1, 2, \ldots, n$), and $|\boldsymbol{x}|_{[i]}$ denotes the i-th smallest component of $|\boldsymbol{x}|$. In other words, $|\boldsymbol{x}|_{[1]}, |\boldsymbol{x}|_{[2]}, \ldots, |\boldsymbol{x}|_{[n]}$ are the absolute values of the components of $\boldsymbol{x}$, sorted in an ascending order. The objective function of problem (5) is essentially based on the order of the absolute component value, which is similar to the modeling method used in Ref. [42]. Then we give the following proposition.

Proposition 1 Let $\boldsymbol{x}^*$ be an optimal solution of problem (5), it holds that,

(1) if $\sum_{i=1}^{n-k} |\boldsymbol{x}^*|_{[i]} > 0$, then problem (4) has no feasible solution;

(2) if $\sum_{i=1}^{n-k} |\boldsymbol{x}^*|_{[i]} = 0$, then $\boldsymbol{x}^*$ is a feasible solution of problem (4); in addition, if any $2k$ columns of the sensing matrix $\boldsymbol{A}$ are linearly independent, then $\boldsymbol{x}^*$ is the unique feasible solution.

It can be seen from Proposition 1 that, the sparse signal can be exactly recovered by solving problem (5) under the assumption that every set of $2k$ columns of the sensing matrix $\boldsymbol{A}$ are linearly independent. Then, we naturally move on to the study of problem (5). First, an introduction to the Continuous Piecewise Linear Function (CPLF) is delivered as follows.

Definition 1 A function $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is called a CPLF if there exists a set of subdomains $\{\Omega_i\}$, where $\bigcup \Omega_i = \Omega$, $\mathring{\Omega}_i \cap \mathring{\Omega}_j = \varnothing, \forall i \neq j$ ($\mathring{\Omega}_i$ is the interior of Ω_i), and f satisfies

(1) $f(\boldsymbol{x}) = f_i(\boldsymbol{x}), \forall \boldsymbol{x} \in \Omega_i$, $f_i(\boldsymbol{x})$ is linear (affine),

(2) $f_i(\boldsymbol{x}) = f_j(\boldsymbol{x}), \forall \boldsymbol{x} \in \Omega_i \cap \Omega_j$.

Proposition 2 Problem (5) can be equivalently transformed into a concave CPLP problem.

Proof Set the index set $I = \{1, 2, \ldots, n\}$, and define $\Theta = \{\theta | \theta \subseteq I, |\theta| = n - k\}$, where $|.|$ denotes the cardinality of the set. Obviously, Θ has $n!/(k!(n-k)!)$ elements. Then, the objective function of problem (5) can be rewritten as follows,

$$f(\boldsymbol{x}) = \sum_{i=1}^{n-k} |\boldsymbol{x}|_{[i]} = \min_{\theta\in\Theta} \sum_{j\in\theta} |x_j| \tag{6}$$

By introducing a new variable $\boldsymbol{u} \in \mathbb{R}^n$, we equally express problem (5) as follows.

$$\begin{aligned} &\min_{\boldsymbol{x},\boldsymbol{u}} \quad f(\boldsymbol{x}, \boldsymbol{u}) = \min_{\theta\in\Theta} \sum_{j\in\theta} u_j, \\ &\text{s.t.} \quad \boldsymbol{Ax} = \boldsymbol{b}, \\ &\qquad \boldsymbol{u} \geqslant \boldsymbol{x}, \\ &\qquad \boldsymbol{u} \geqslant -\boldsymbol{x} \end{aligned} \tag{7}$$

For simplicity, let

$$\boldsymbol{z} = (u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^{2n},$$

and problem (7) can be equivalently transformed into the following form,

$$\begin{aligned} \min_{z} \quad & f(z) = \min_{\theta \in \Theta} \textstyle\sum_{j\in\theta} b_j, \\ \text{s.t.} \quad & \hat{\boldsymbol{A}} z = \boldsymbol{b}, \\ & \boldsymbol{C} z \leqslant \boldsymbol{0} \end{aligned} \tag{8}$$

where

$$\hat{\boldsymbol{A}} = (\boldsymbol{0}_{m\times n}|\boldsymbol{A}), \boldsymbol{C} = \begin{pmatrix} -\boldsymbol{I}_{n\times n} & \boldsymbol{I}_{n\times n} \\ -\boldsymbol{I}_{n\times n} & -\boldsymbol{I}_{n\times n} \end{pmatrix}.$$

Function $f(z)$ is concave on account of that it is actually the pointwise minimum of $M = n!/(k!(n-k)!)$ linear functions.

Denote Θ by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_M\}$. We define the corresponding subdomains $\Omega_{\theta_i} \subseteq \mathbb{R}^{2n}$ as

$$\begin{aligned} \Omega_{\theta_i} = \{z \in \mathbb{R}^{2n} | z_s \leqslant z_t, \forall s \in \theta_i, \forall t \in I\backslash\theta_i\}, \\ i \in \{1, 2, \ldots, M\} \end{aligned} \tag{9}$$

which satisfies

$$\begin{aligned} &\mathbb{R}^{2n} = \textstyle\bigcup_{i=1}^{M} \Omega_{\theta_i}, \\ &\mathring{\Omega}_{\theta_i} \cap \mathring{\Omega}_{\theta_j} = \varnothing, \forall i \neq j \end{aligned} \tag{10}$$

Notably, it holds that

$$\begin{aligned} &f(z) = f_{\theta_i}(z) = \sum_{j\in\theta_i} z_j, \forall z \in \Omega_{\theta_i}, \\ &f_{\theta_i}(z) = f_{\theta_j}(z), \forall z \in \Omega_{\theta_i} \cap \Omega_{\theta_j} \end{aligned} \tag{11}$$

Therefore, $f(z)$ is a CPLF according to Definition 1, and problem (8) is a concave CPLP problem on a convex polyhedron domain. □

For Proposition 2, notice that

(1) Any point $z \in \mathbb{R}^{2n}$ must be located in at least one subdomain.

(2) If $z \in \mathbb{R}^{2n}$ is located in several subdomains, it must be a boundary point of these subdomains.

(3) Further, rearrange the first n entries of $z \in \mathbb{R}^{2n}$ in an ascending order, there have one or more possible orders (due to some entries may be equal in magnitude). Arbitrarily pick one possible order denoted by $z_{i_1} \leqslant z_{i_2} \leqslant \cdots \leqslant z_{i_{n-k}} \leqslant \cdots \leqslant z_{i_n}$, then we can determine an index set $\Phi_z \in \Theta$ as

$$\Phi_z = \{i_1, i_2, \ldots, i_{n-k}\} \tag{12}$$

which satisfies $z \in \Omega_{\Phi_z}$ (i.e., Φ_z is identical to a certain $\theta_i, i \in \{1, 2, \ldots, M\}$). Obviously, different possible orders correspond to different subdomains, respectively.

(4) Therefore, fix $z \in \mathbb{R}^{2n}$, and we can determine the subdomain Ω_{Φ_z} which contains z, then we can obtain the corresponding linear fragments $f(\boldsymbol{x}) = f_{\Phi_z}(z), \forall z \in \Omega_{\Phi_z}$.

Theoretically speaking, by obtaining the optimum of problem (8), we can reconstruct the sparse signal according to Proposition 1. Thus, the crux of this paper is to design such an optimization algorithm for problem (8).

3 CS-IPM Algorithm

The above discussion connects conventional sparse signal reconstruction in CS to concave piecewise linear programming. In order to solve it, we employ a modified interior point method[43] called CS-IPM. In this section, an integrated and detailed description of this algorithm will be given, including algorithm procedures, technical analysis, and implementation details.

3.1 Algorithmic framework

Denote the feasible domain of problem (8) by Ω. Given a feasible solution $\hat{z} \in \mathbb{R}^{2n}$, we are able to find the sub-region $\Omega_{\Phi_{\hat{z}}} \in \Omega$, where the realization of $f(z)$ is linear. Denote this realization of $f(z)$ by $f_{\hat{z}}(z) = \boldsymbol{g}_{\hat{z}}^{\mathrm{T}} z$. Therefore, problem (8) is reduced to the following LP problem,

$$\begin{aligned} \min_{z} \quad & f_{\hat{z}}(z) = \boldsymbol{g}_{\hat{z}}^{\mathrm{T}} z, \\ \text{s.t.} \quad & \hat{\boldsymbol{A}} z = \boldsymbol{b}, \\ & \boldsymbol{C} z \leqslant 0 \end{aligned} \tag{13}$$

Then the KKT condition for the global optimum is as follows,

$$\begin{cases} \boldsymbol{g}_{\hat{z}} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\xi} + \hat{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{\zeta} = 0, \\ \operatorname{diag}(\boldsymbol{\xi})\boldsymbol{C} z = 0, \\ \boldsymbol{C} z \leqslant 0, \\ \hat{\boldsymbol{A}} z = \boldsymbol{b}, \\ \boldsymbol{\xi} \geqslant 0 \end{cases} \tag{14}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are dual variables and $\operatorname{diag}(\boldsymbol{\xi})$ is the diagonal matrix whose diagonal is $\boldsymbol{\xi}$.

To use the interior point method, we modify the second equation in problem (14) such that

$$\operatorname{diag}(\boldsymbol{\xi})\boldsymbol{C} z = -\boldsymbol{1}\sigma\boldsymbol{\mu} \tag{15}$$

where $\boldsymbol{\mu} = -(\boldsymbol{C} z)^{\mathrm{T}}\boldsymbol{\xi}/m$ is the duality measure and $\sigma \in [0, 1]$. The main concept of the interior point method is to solve the KKT conditions (14) with constantly updated $\boldsymbol{\mu}$ and σ until the algorithm converges.

By utilizing the Taylor series, we can obtain the following Newton equations,

$$\begin{pmatrix} 0 & \boldsymbol{C}^{\mathrm{T}} & \hat{\boldsymbol{A}}^{\mathrm{T}} \\ \hat{\boldsymbol{A}} & 0 & 0 \\ \operatorname{diag}(\boldsymbol{\xi})\boldsymbol{C} & \operatorname{diag}(\boldsymbol{C} z) & 0 \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta\boldsymbol{\zeta} \\ \Delta\boldsymbol{\xi} \end{pmatrix} = -\begin{pmatrix} \boldsymbol{r}_d \\ \boldsymbol{r}_p \\ \boldsymbol{r}_c \end{pmatrix} \tag{16}$$

where $\operatorname{diag}(\boldsymbol{C} z)$ is the diagonal matrix with $(\boldsymbol{C} z)$ as its main diagonal and the right-hand side is defined as the dual, primal and central residual of the solution $\hat{z}$, respectively,

$$\begin{cases} \boldsymbol{r}_d = \boldsymbol{g}_{\hat{z}} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\xi} + \hat{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{\zeta}, \\ \boldsymbol{r}_p = \hat{\boldsymbol{A}}z - \boldsymbol{b}, \\ \boldsymbol{r}_c = -\mathbf{1}\sigma\boldsymbol{\mu} - \mathrm{diag}(\boldsymbol{\xi})\boldsymbol{C}z \end{cases} \tag{17}$$

From the assumption of the feasibility of the starting point, we know that $\boldsymbol{r}_p = 0$. Then the interior point method can be applied and we can retain a converged solution for LP (13). However, if a complete interior point method procedure is conducted, the eventually obtained solution could not be guaranteed to be globally optimal or even locally optimal to problem (8), due to the non-linearity of $f(z)$.

A potential solution is to only do the update procedure according to Eq. (16) once, and then check if the obtained solution $\bar{z} = \hat{z} + \Delta z$ lies in $\Omega_{\Phi_{\hat{z}}}$. If yes, then remain the objective function in problem (13) unchanged; otherwise, update the value of $\boldsymbol{g}_{\hat{z}}$ in problem (13) with $\boldsymbol{g}_{\bar{z}}$, which corresponds to the linear realisation of $f(z)$ in the sub-region where $\bar{z}$ belongs to. Sequentially, iteratively repeat the above procedures to find a solution in the feasible domain.

3.2 Algorithm scheme

In summary, the detailed implementation of this modified interior point method is presented in Algorithm 1.

Algorithm 1 CS-IPM

Require:
- Signal sparsity k;
- Measurement matrix $\boldsymbol{A}$;
- Observations $\boldsymbol{b}$.

Ensure:
- Recovered signal $\boldsymbol{x}$.

Build the concave CPLP model as (8) using $k, \boldsymbol{A}, \boldsymbol{b}$;
Randomly generate a feasible starting point $(z^{(0)}, \boldsymbol{\zeta}^{(0)}, \boldsymbol{\xi}^{(0)})$, and find its linear realization $f_{z^{(0)}}$;
Let $g^{(0)} := \nabla f_{z^{(0)}}$ and $k = 0$;
while $\boldsymbol{\mu}^k > \varepsilon$ or $g^{(k)} \neq g^{(k-1)}$ or $||\boldsymbol{r}_{\mathrm{dual}}|| > \epsilon$ **do**
  Solve the equations in (16)(with $\sigma = 0$ if k is even, and $\sigma = 1$ if k is odd), and obtain the Newton direction $(\Delta z^{(k)}, \Delta\boldsymbol{\zeta}^{(k)}, \Delta\boldsymbol{\xi}^{(k)})$;
  Do line searching along $(\Delta z^{(k)}, \Delta\boldsymbol{\zeta}^{(k)}, \Delta\boldsymbol{\xi}^{(k)})$ such that
    i) $-(\boldsymbol{C}(z^{(k)} + \alpha_k\Delta z))^{\mathrm{T}}(\boldsymbol{\xi}^{(k)} + \alpha_k\Delta\boldsymbol{\xi}) \leqslant \boldsymbol{\mu}^{(k)}$;
    ii) $(z^{(k)} + \alpha_k\Delta z, \boldsymbol{\zeta}^{(k)} + \alpha_k\Delta\boldsymbol{\zeta}, \boldsymbol{\xi}^{(k)} + \alpha_k\Delta\boldsymbol{\xi})$ lies in $\mathcal{N}_2(\theta) = \{(z, \boldsymbol{\zeta}, \boldsymbol{\xi}) \mid ||\boldsymbol{C}z\boldsymbol{\xi} - \boldsymbol{\mu}\mathbf{1}||_2 \leqslant \theta\boldsymbol{\mu}\}$;
  where α_k is the step length and $\theta = 0.1$ when k is even, and $\theta = 0.235$ otherwise;
  Update
$$\begin{cases} z^{(k+1)} := z^{(k)} + \alpha_k\Delta z, \\ \boldsymbol{\zeta}^{(k+1)} := \boldsymbol{\zeta}^{(k)} + \alpha_k\Delta\boldsymbol{\zeta}, \\ \boldsymbol{\xi}^{(k+1)} := \boldsymbol{\xi}^{(k)} + \alpha_k\Delta\boldsymbol{\xi}, \\ \boldsymbol{\mu}^{(k+1)} := -(\boldsymbol{C}(z^{(k)} + \alpha_k\Delta z))^{\mathrm{T}}(\boldsymbol{\xi}^{(k)} + \alpha_k\Delta\boldsymbol{\xi}); \end{cases} \tag{18}$$
  Find the corresponding realization $f_{z^{(k+1)}}$ and let $g^{(k+1)} = \nabla f_{z^{(k+1)}}$;
  Set $k := k + 1$;
end while
Obtain the optimum $z^* := z^{(k)}$;
Return $\boldsymbol{x} := z^*(n + 1 : 2n)$.

4 Numerical Experiments

In this section, CS-IPM is compared to four representatives: BP, IRLS, ISD, and CoSaMP. Among these, ISD and IRLS appear to be state-of-the-art in terms of the number of measurements required. The comparisons show that CS-IPM requires as few measurements as them but has a high processing speed.

4.1 Experimental settings and test platforms

We conduct five kinds of experiments according to the different experimental purposes, and all the settings are summarized in Table 1. The first three experiments use various synthetic standard i.i.d. Gaussian signals and sensing matrices $\boldsymbol{A}$. Experiments 1 and 2 use noise-free measurements with diverse problem scales. As we know, the measurement data is not always accurate on account of various kinds of imprecisions or interferences. We contaminate our data with white Gaussian noise and the observations $\boldsymbol{b}$ obey $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{e}$, where $\boldsymbol{e}$ denotes an i.i.d. $N(0, \sigma^2)$ noise. Exact reconstruction is impossible in this noisy scenario, so in Experiment 3 we keep m unchanged but measure solution relative errors for four levels of noise. The fourth experiment tries another type of signal, sparse Bernoulli signal, which only takes ± 1 as nonzeros. Experiment 5 replaces the Gaussian sensing matrix by badly-conditioned matrix to test its impact.

All codes are written and tested in Matlab 2013a. Computations are conducted on a Windows machine with Core i3 3.30 GHz processor and 8 GB of RAM.

4.2 Experimental results

4.2.1 Experiment 1

This experiment is to determine how many measurements m are necessary to recover a k-sparse signal in $\mathbb{R}^n$ with a high probability via different algorithms respectively. We use synthetic sparse signals and standard i.i.d. Gaussian sensing matrices. The success criteria of reconstruction is based upon

$$\rho = \frac{||\boldsymbol{x} - \bar{\boldsymbol{x}}||_2}{||\bar{\boldsymbol{x}}||_2} \leqslant 10^{-6} \tag{19}$$

Sparse signals containing $k = 5, 50, 100, 200, 300$ nonzeros are used in this experiment, and the

Table 1 Summary of experimental settings.

No.	Signal	Sensing matrix	Noise σ	Dimension n	Sparsity k	Measurements m
1	Gaussian	Gaussian	0	500	5	10:5:40
	Gaussian	Gaussian	0	500	50	100:50:250
	Gaussian	Gaussian	0	500	100	200:100:400
2	Gaussian	Gaussian	0	2000	100	200:600
3	Gaussian	Gaussian	0.1	500	50	200
	Gaussian	Gaussian	0.01	500	50	200
	Gaussian	Gaussian	0.001	500	50	200
	Gaussian	Gaussian	0.0001	500	50	200
4	Bernoulli	Gaussian	0	500	5	10:5:35
	Bernoulli	Gaussian	0	500	50	100:50:350
	Bernoulli	Gaussian	0	500	100	200:100:400
5	Gaussian	cond=3	0	100	5	10:5:30
	Gaussian	cond=100	0	100	5	10:5:30
	Gaussian	cond=1000	0	100	5	10:5:30

corresponding results are plotted in Figs. 1–5, respectively. They display the correct reconstruction percentage or computational time as a function of m. Each curve represents the performance of an algorithm.

Figure 1 depicts the performance of five tested algorithms under a extremely sparse condition. As expected, for a fixed sparsity level, the percentages increase as we take more measurements. CS-IPM achieves a recoverability close to ISD, which is much higher than that of the others in terms of the number of required measurements. Meanwhile, CS-IPM has a significantly faster speed than ISD, and is even comparable to BP.

With the sparsity $k = 50$ in Fig. 2, IRLS and ISD have better performance when $m \leqslant 190$. However, when $m > 190$, CS-IPM performs best and achieves the reconstruction percentage with 100% starting around $m = 200$. In terms of the computational time, CoSaMP is much faster than the others.

With the sparsity $k = 100$ in Fig. 3, CS-IPM has no speed advantage over ISD, but it is still faster than IRLS when m is small. Qualitywise, CS-IPM is on par with IRLS and better than ISD.

Continue to increase k, we test their performance in a less sparse condition. With the sparsity $k = 200$ in Fig. 4, CS-IPM and BP have the same recoverability which is much higher than the other three. ISD becomes less effective in this case. In terms of the computational time, CS-IPM is faster than BP.

When the sparsity k increases to 300, CS-IPM, BP, and IRLS achieve the reconstruction percentage with 100% starting around $m = 500$ ($m = 1.7k$). However, ISD and CoSaMP are entirely ineffective even when m increases to 500 (see Fig. 5).

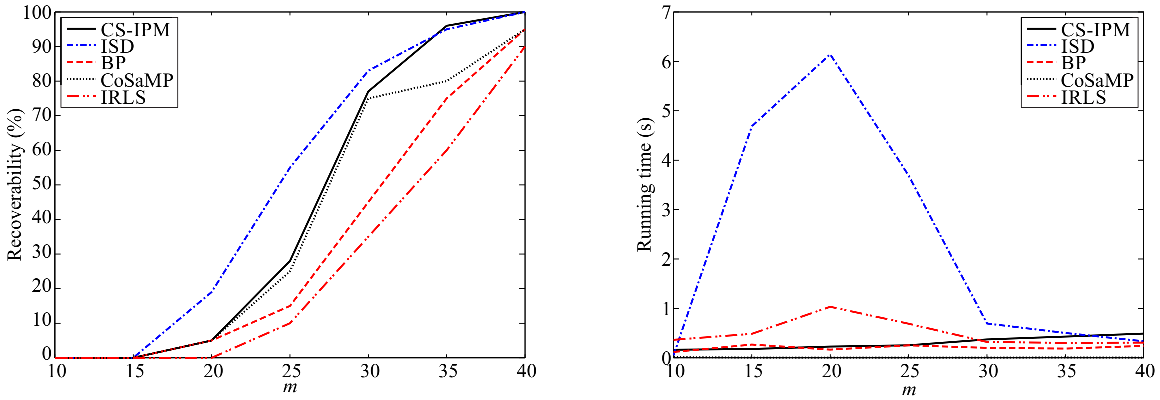
Table 2 presents another view of the same data. It demonstrates the ratios of the sufficient number of measurements for exact reconstruction against the sparsity level m/k under various settings. The symbol "/" means the recovery percentage maintains 0% even when $m = n$. We discover that the sparsity of the signal has a great impact on the performance that sparser signal needs more measurements. Overall, CS-IPM is superior to the others.

The above results are based on the success criteria (19). Many fast algorithms do not pursue a high accuracy but have good performance for a low accuracy. Therefore, we relax the criteria from 10^{-6} to 10^{-3} and test the performance for the purpose of application.

As shown in Figs. 6 and 7, when ρ reduces to 10^{-3}, all algorithms have the higher calculating speed. IRLS and ISD have the better performance than the other three. Hence, they are more suitable for some practical uses, which require less accuracy.

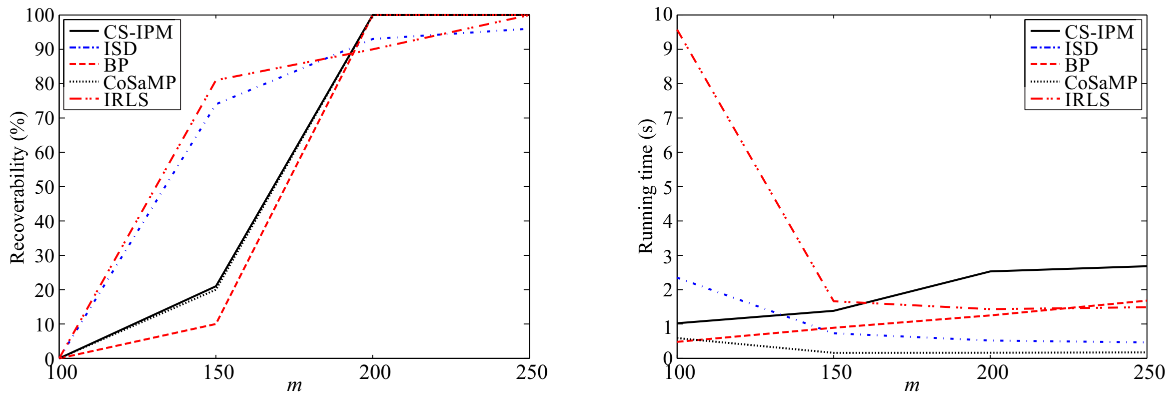
Table 2 Ratio of the sufficient number of measurements for exact reconstruction against the sparsity level.

Algorithm	m/k				
	$k = 5$	$k = 50$	$k = 100$	$k = 200$	$k = 300$
CS-IPM	8	4	3	2	1.7
IRLS	10	5	3	2.5	1.7
ISD	8	6	4.4	/	/
CoSaMP	9.2	4	3	2.5	/
BP	9	4	3	2	1.7



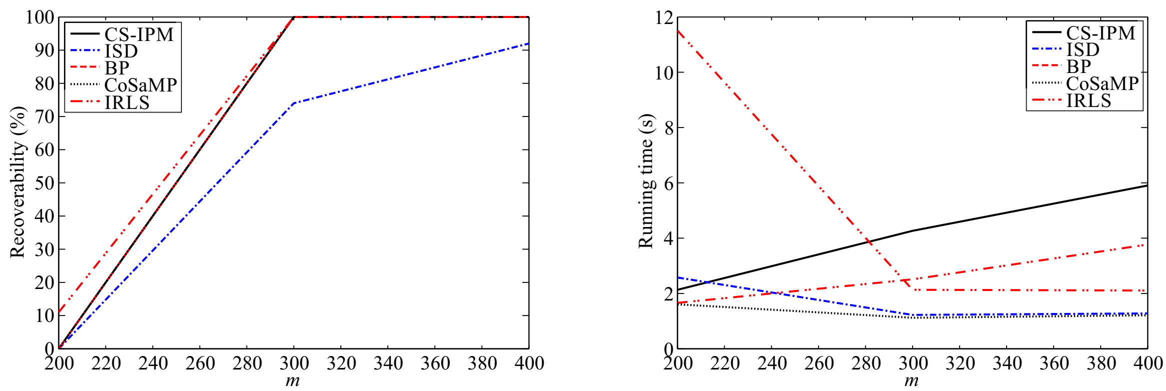
(a) (b)

Fig. 1 Comparisons in recoverability and CPU time under setting 1 (*n*=500, *k*=5).



(a) (b)

Fig. 2 Comparisons in recoverability and CPU time under setting 2 (*n*=500, *k*=50).



(a) (b)

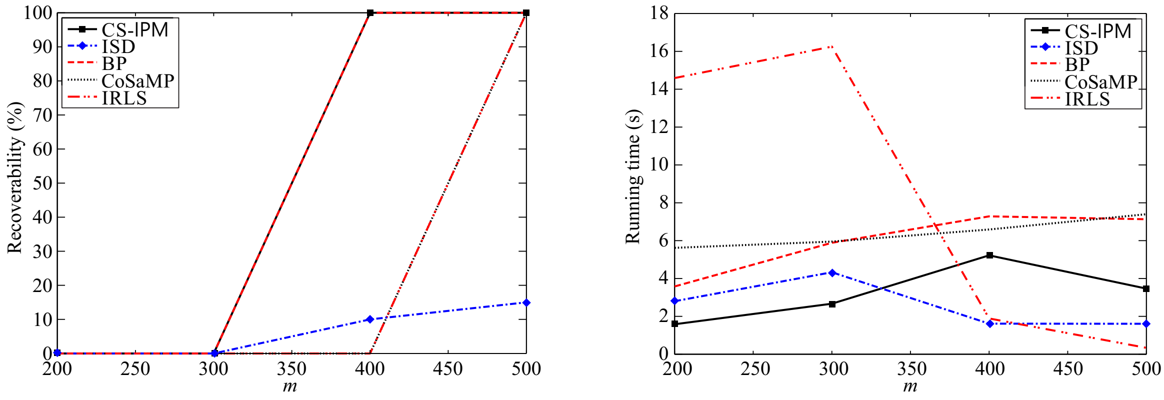
Fig. 3 Comparisons in recoverability and CPU time under setting 3 (*n*=500, *k*=100).

4.2.2 Experiment 2

In this experiment we test algorithms on larger signals ($n = 2000$). Figure 8 shows that algorithms perform similarly to what they did in lower dimension. Not surprisingly, CoSaMP is the fastest one. Qualitywise, CS-IPM and IRLS have the best recoverability, and achieves exact reconstruction percentage with 100% starting around $m = 5k$. These results also show that the propose algorithm is scalable to both signal and measurements sizes.

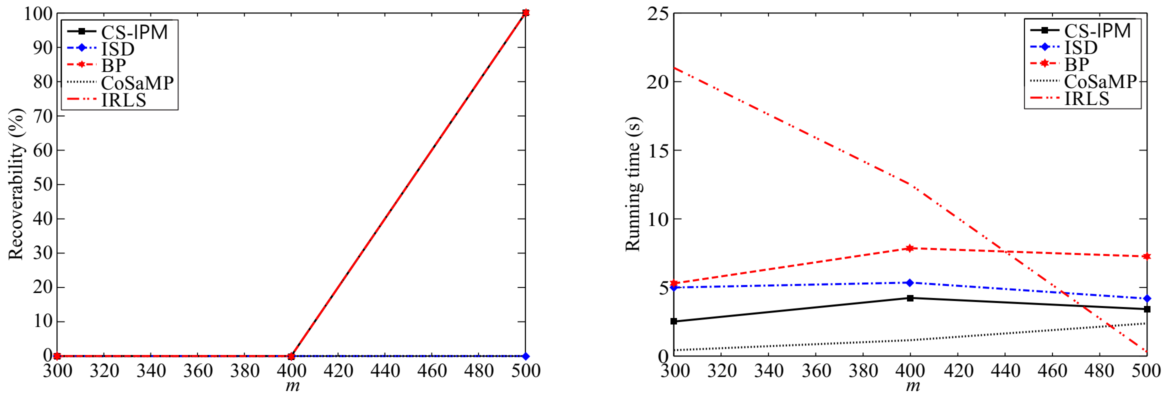
4.2.3 Experiment 3

This experiment compares the performance of the tested algorithms given noisy measurements. With noise, exact recovery is impossible. Thus we fix the problem



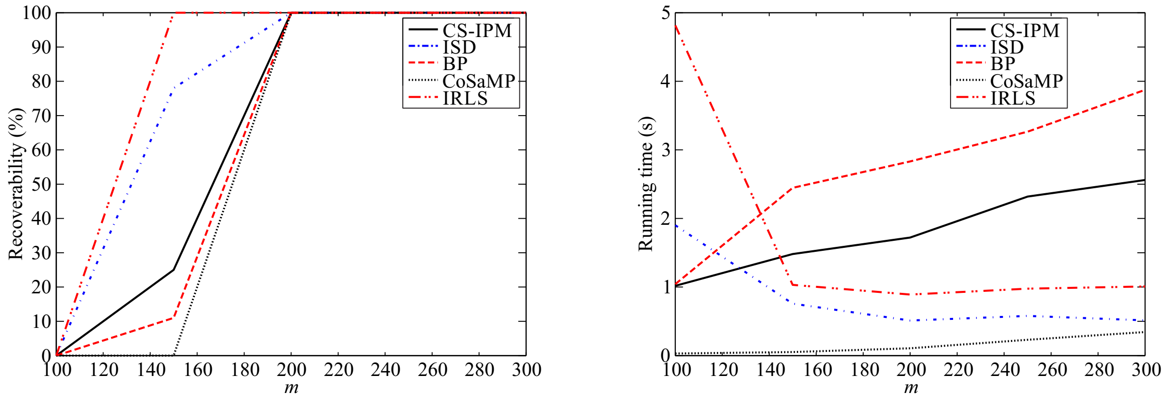
(a) (b)

Fig. 4 Comparisons in recoverability and CPU time under setting 4 (n=500, k=200).



(a) (b)

Fig. 5 Comparisons in recoverability and CPU time under setting 5 (n=500, k=300).



(a) (b)

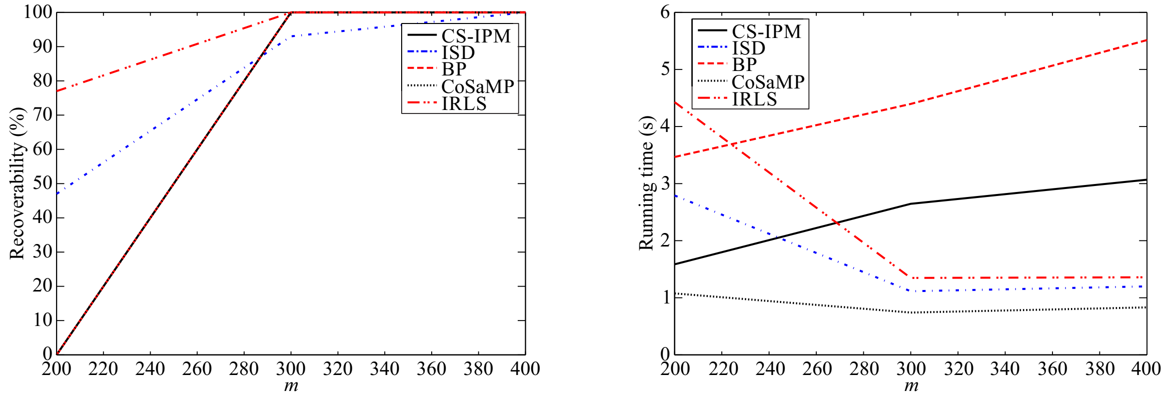
Fig. 6 Comparisons with ρ=10^{-3} under setting 1 (n=500, k=50).

scale and measure the solution relative errors with four noise levels: $\sigma \in \{0.1, 0.01, 0.001, 0.0001\}$. The corresponding results are depicted in subplots of Fig. 9.

It is clear from the figure that the errors are proportional to σ. When σ is big, ISD has the smallest relative errors while BP has the biggest. However, when σ is small, ISD has the worst anti-noise ability, and CS-IPM, BP, IRLS, and CoSaMP are on par in solution quality.

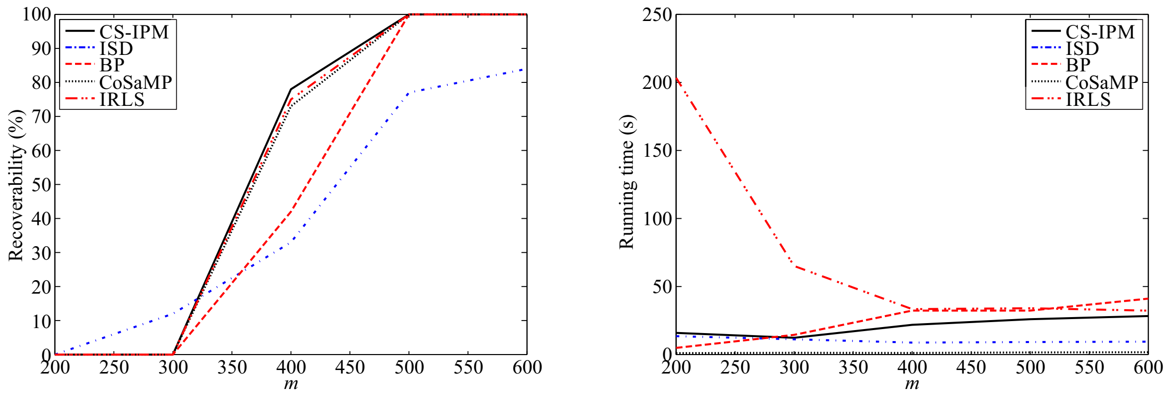
4.2.4 Experiment 4

This experiment changes the Gaussian signals to the sparse Bernoulli signals, and the corresponding results are plotted in Figs. 10–12. Compare with the results



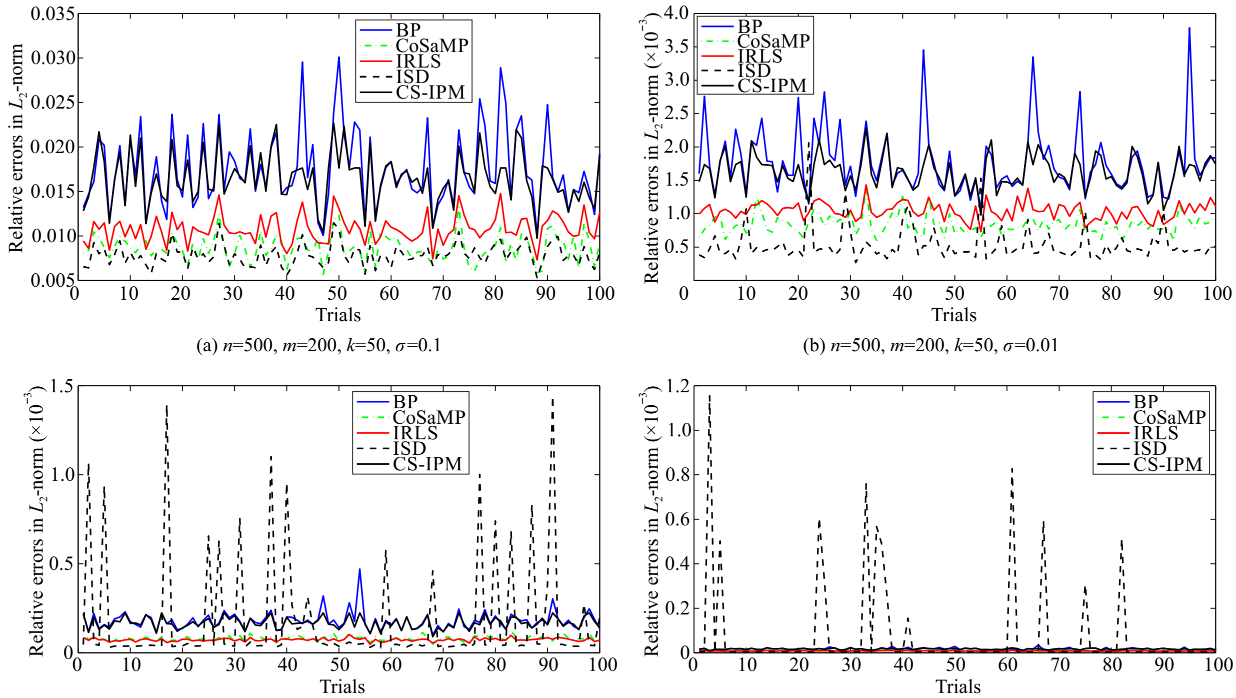
(a) (b)

Fig. 7 Comparisons with ρ=10^{-3} under setting 2 (n=500, k=100).



(a) (b)

Fig. 8 Comparisons in recoverability and CPU time with lager signals (n=2000, k=100).



(a) n=500, m=200, k=50, σ=0.1
(b) n=500, m=200, k=50, σ=0.01
(c) n=500, m=200, k=50, σ=0.001
(d) n=500, m=200, k=50, σ=0.0001

Fig. 9 Comparisons in reconstruction errors.

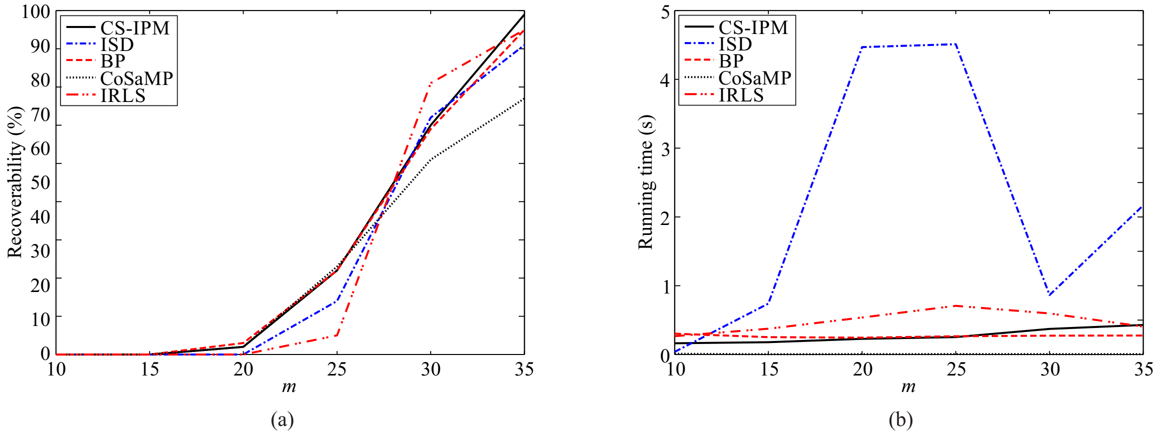


Fig. 10 Comparisons in recoverability and CPU time with Bernoulli signals under setting 1 (*n*=500, *k*=5).

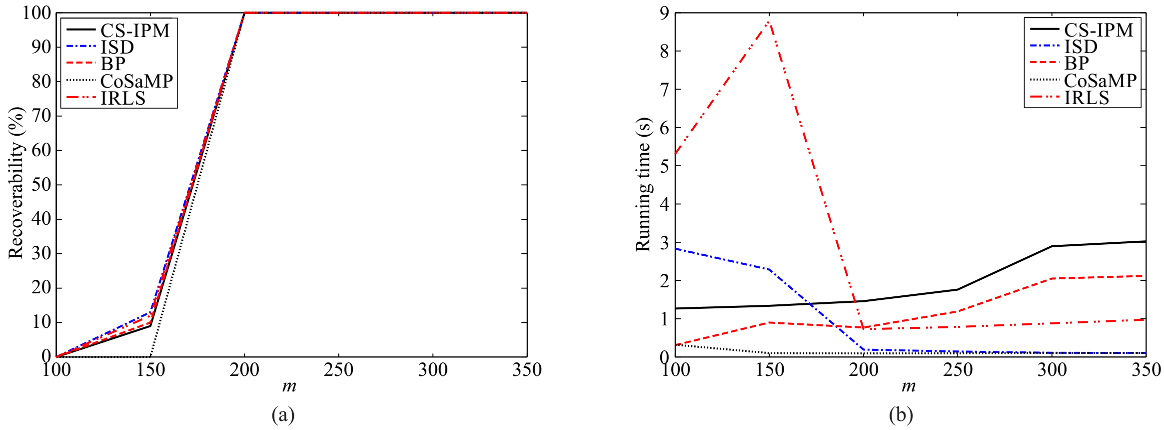


Fig. 11 Comparisons in recoverability and CPU time with Bernoulli signals under setting 2 (*n*=500, *k*=50).

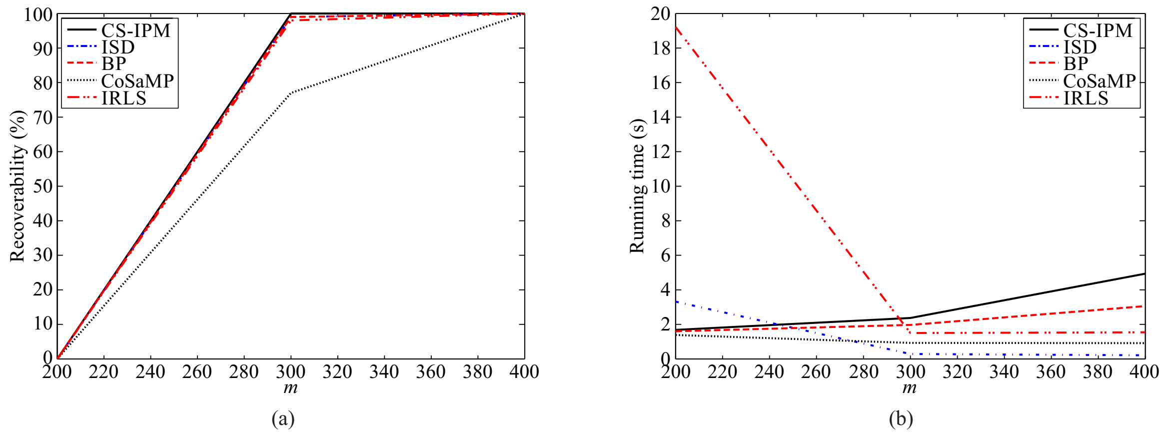


Fig. 12 Comparisons in recoverability and CPU time with Bernoulli signals under setting 3 (*n*=500, *k*=100).

of Experiment 1, CS-IPM, BP, IRLS, and ISD have similar recoverability and computational time. Although CoSaMP is still the fastest, changed signal type results in decreased recoverability. Taking both recoverability and computational time into consideration, CS-IPM is the best.

4.2.5 Experiment 5

This experiment discusses the computational performance under the influence of the property of the sensing matrix.

Conventional algorithms need the sensing matrix to meet some special demands to guarantee the robust recovery. As stated in Ref. [3], Candes and Tao proposed the RIP condition that the geometry of the sparse signals should be preserved under the action of the sensing matrix. To quantify this idea, they defined the r-th restricted isometry constant of $\boldsymbol{A}$ as the least number δ_r, for which

$$(1-\delta_r)\ \|\boldsymbol{x}\|_2^2 \leqslant \|\boldsymbol{Ax}\|_2^2 \leqslant (1+\delta_r)\ \|\boldsymbol{x}\|_2^2 \qquad (20)$$

whenever $\|\boldsymbol{x}\|_0 \leqslant r$. Herein, δ_r quantifies how close to isometrically $\boldsymbol{A}$ acts on r-sparse signals. When $\delta_r \ll 1$, the sampling operator nearly maintains the distance between each pair of $(r/2)$-sparse signals. Consequently, the RIP condition requires a very small δ_r for the stable reconstruction.

This actually implies that, the algorithms based on RIP or quasi-RIP require a small condition number of $\boldsymbol{A}$, which is denoted by cond($\boldsymbol{A}$) and defined as

$$\text{cond}(\boldsymbol{A}) = \lambda_{\max}(\boldsymbol{A})/\lambda_{\min}(\boldsymbol{A}) \qquad (21)$$

where $\lambda_{\max}(\boldsymbol{A})$ and $\lambda_{\min}(\boldsymbol{A})$ represent the maximal and minimal singular values of $\boldsymbol{A}$, respectively. By contrast, CS-IPM does not require this property according to the previous analysis. Therefore, we use different sensing matrices with respect to the varying cond($\boldsymbol{A}$) to test their impact on the computational performance.

Let $n = 100$, $k = 5$, $m \in \{10, 11, \cdots, 30\}$. The contrast experiments are conducted with different condition number: cond($\boldsymbol{A}$) $\in \{3, 100, 1000\}$. For each condition number, we perform 100 independent trials adopting CS-IPM, CoSaMP (RIP-based), and ISD (quasi-RIP-based), respectively.

As expected, ill-conditioned sensing matrices restrict the performances of CoSaMP and ISD. Additionally, this disadvantage becomes more obvious as the condition number increases. However, the proposed algorithm CS-IPM performs stably, no matter with what condition number, which is illustrated in Fig. 13.

To sum up Experiments 1–5, we state that CS-IPM

- can effectively recover the sparse signal with the least sufficient measurements to our knowledge;
- performs similarly with Gaussian and Bernoulli signals;
- is relatively stable in the noisy scenarios, however, when σ gets bigger, all the tested algorithms are known to yield the worse solutions;
- does not require the additional restrictions for the sensing matrix like RIP. It performs robustly to the dramatic increase of the condition number of the sensing matrix.

5 Conclusion

This paper introduces the CS-IPM algorithm for the sparse signal reconstruction, which is based on a concave continuous piecewise linear programming model. Both theoretical and practical performances are discussed. CS-IPM makes use of the concavity and piecewise linearity of the proposed model to implement the descent process. Additionally, we incorporate a modified interior point method to obtain the optima. The new framework benefits us with few number of measurements and short CPU time. In the computational experiments, the good performance of our proposed algorithm is confirmed.

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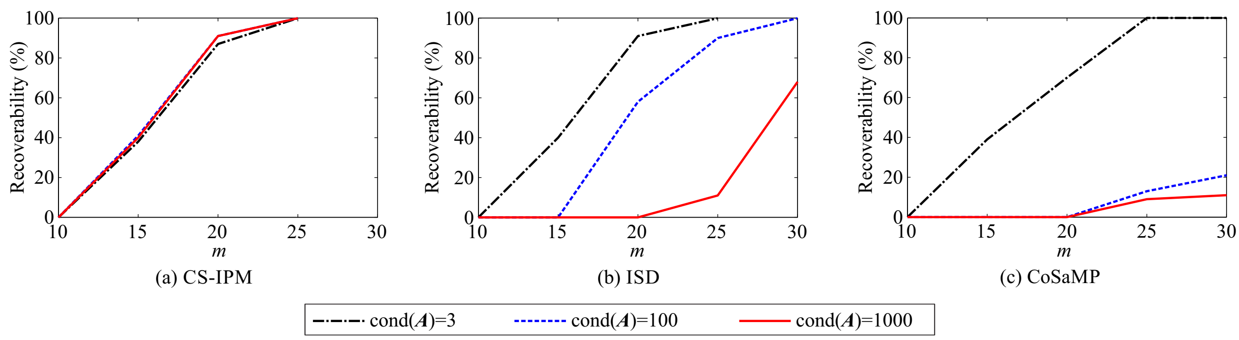

Fig. 13 The influence of increasing condition numbers.

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