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Consensus of Second-Order Multi-Agent Systems with Time-Varying Delays and Antagonistic Interactions

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Abstract: This study investigates the consensus problem of second-order multi-agent systems subject to time-varying interval-like delays. The notion of consensus is extended to networks containing antagonistic interactions modeled by negative weights on the communication graph. A unified framework is established to address both the stationary and dynamic consensus issues in sampled-data settings. Using the reciprocally convex approach, a sufficient condition for consensus is derived in terms of matrix inequalities. Numerical examples are provided to illustrate the effectiveness of the proposed result.

Key words: multi-agent systems; second-order; consensus; cooperation and competition

1 Introduction

Multi-Agent Systems (MAS) possess great superiority to traditional monolithic systems in terms of reliability, rescalablity, and flexibility. In the last two decades, MAS-related studies, particularly those on consensus, have attracted considerable attention from various scientific communities.

Time delays are ubiquitous in real-world systems, and many remarkable studies have been conducted on delayed MAS. To name a few, Olfati-Saber and Murray's work proved that for the first-order MAS, the upper bound of allowable delay is inversely proportional to the maximum degree of the communication topology[1]. It was shown in Ref. [2] that the consensus condition is dependent on input delays, yet independent of communication delays. By exploring the properties of non-negative matrices, consensus problems for delayed first- and second-order MAS were studied in Refs. [3, 4], respectively. Yu et al.[5] provided a necessary and sufficient condition for consensus of second-order MAS with fixed topology and uniform delay. An upper bound of allowable delay contacted with the eigenvalues of the Laplacian matrix was given. The non-uniform delay case was addressed in Ref. [6] using a frequency approach.

Note that, most existing studies focused on determining an upper bound of the allowable time delay[7]. The derived criteria for consensus are delay-independent, which usually possess more conservativeness than delay-dependent ones[8]. In addition, in many circumstances, time-delay is within an interval, i.e., it is upper and lower bounded. On the other hand, one fundamental assumption made by these works on consensus is that all agents are cooperative. In real-world scenarios, non-cooperative or even antagonistic interactions also exist, particularly in contexts such as markets or social networks. It is more challenging, yet of great interest to study the consensus problem in non-cooperative systems.

This paper aims to address the consensus problems for second-order MAS subject to antagonistic interactions and time-varying interval-like input delays. The antagonistic interactions are modeled by negative weights on the signed graph[9–12]. Thus, the methods used in many studies that explored the

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non-negative properties of Stochastic, Indecomposable and Aperiodic (SIA) matrices[3, 4, 13, 14] no long apply in this scenario. By constructing a Lyapunov-Krasovskii functional, a delay-dependent sufficient condition for consensus is derived in terms of Linear Matrix Inequalities (LMIs), using the reciprocally convex approach[15, 16]. Both stationary and dynamic consensus cases are considered and addressed within a unified framework.

The remainder of this paper is organized as follows. In Section 2, some preliminaries on the signed graph are given. In Section 3, the problems to be investigated are formulated in detail. The main result is illustrated in Section 4. Both stationary and dynamic consensus examples validating the effectiveness of the results are provided in Section 5. Finally, concluding remarks are stated in Section 6.

Notations: Throughout the paper, $\boldsymbol{I}$ and $\boldsymbol{0}$ stand for the identity and zero matrices of appropriate dimensions, respectively. The superscript "T" represents matrix transpose. We use an asterisk "*" within large matrices to represent a block induced by symmetry. diag$\{\cdots\}$ stands for a block diagonal matrix. A matrix $\boldsymbol{\Omega} > 0$ if and only if $\boldsymbol{\Omega}$ is symmetric and positive definite. If the dimension of a matrix is not denoted explicitly, it is assumed to be compatible with algebraic operations.

2 Preliminaries

For a system of N agents in the cooperation-competition network, its communication link topology is modeled as a signed and directed graph denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{W}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of nodes representing agents, and $\mathcal{W} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The directed link (v_i, v_j) enables agent j to access the state information of agent i. The adjacent matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is defined as $a_{ij} \neq 0$ if $(v_j, v_i) \in \mathcal{W}$, otherwise $a_{ij} = 0$. Different from assumptions in the cooperative consensus case, the matrix weight of the coupling graph in the cooperation-competition network is not assumed to be nonnegative. A negative weight a_{ij} means agent j interacts with agent i antagonistically, which prevents rather than promotes consensus. The Laplacian matrix $\boldsymbol{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ for graph $\mathcal{G}$ is defined as $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$.

Two assumptions on the system topology and delay respectively are given below.

Assumption 1 The communication network for the system is fixed, and there exists at least one cooperative link between any agent and its neighbors.

Remark 1 For a cooperative network, a directed spanning tree can be a necessary condition for consensus. In this paper, we consider the extra antagonistic interactions, which prevent rather than promote consensus. If one agent only interacts antagonistically with its neighbors, there is no chance for this agent to agree with others to achieve consensus.

Assumption 2 The time delay τ_k at instance k is identical all over the system and satisfies $0 < \tau_m \leqslant \tau_k \leqslant \tau_M$, where τ_m and τ_M are positive integers.

Lemma 1 For any matrix $\boldsymbol{M}$ satisfying $\boldsymbol{M} > 0$ and $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, integers $k_2 \geqslant k_1$, and vector function $\boldsymbol{\omega}(i) \in \mathbb{R}^n$, the following inequality holds[17]:

$$(k_2 - k_1 + 1) \sum_{i=k_1}^{k_2} \boldsymbol{\omega}^{\mathrm{T}}(i)\boldsymbol{M}\boldsymbol{\omega}(i) \geqslant \left(\sum_{i=k_1}^{k_2} \boldsymbol{\omega}(i)\right)^{\mathrm{T}} \boldsymbol{M} \left(\sum_{i=k_1}^{k_2} \boldsymbol{\omega}(i)\right) \quad (1)$$

Lemma 2 Reciprocally convex approach[15]: For vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, symmetric matrix $\boldsymbol{\Omega}$, any matrix $\boldsymbol{\Psi}$, and a positive real number ε satisfying $\begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\Psi} \\ \boldsymbol{\Psi}^{\mathrm{T}} & \boldsymbol{\Omega} \end{bmatrix} \geqslant 0$, if $\varepsilon \neq 1$,

$$\begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Omega} & \boldsymbol{\Psi} \\ \boldsymbol{\Psi}^{\mathrm{T}} & \boldsymbol{\Omega} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{bmatrix} \leqslant \varepsilon^{-1}\boldsymbol{\eta}_1^{\mathrm{T}}\boldsymbol{\Omega}\boldsymbol{\eta}_1 + (1-\varepsilon)^{-1}\boldsymbol{\eta}_2^{\mathrm{T}}\boldsymbol{\Omega}\boldsymbol{\eta}_2.$$

3 Problem Formulation

We consider the second-order MAS in the sampled-data setting. The dynamics of the agent i can be modeled as follows:

$$\begin{cases} x_i(k+1) = x_i(k) + hv_i(k) + \dfrac{h^2}{2}u_i(k), \\ v_i(k+1) = v_i(k) + hu_i(k) \end{cases} \quad (2)$$

where $x_i \in \mathbb{R}$, $v_i \in \mathbb{R}$, and $u_i \in \mathbb{R}$ are the position, velocity, and control input of agent i, respectively, $i = 1, 2, \cdots, N$, and $h \in \mathbb{R}^+$ is the sampling period.

From Eqs. (2), we can obtain

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{v}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & h\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{v}(k) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2}\boldsymbol{I} \\ h\boldsymbol{I} \end{bmatrix} \boldsymbol{u}(k) \quad (3)$$

with

$$\boldsymbol{x}(k) = [x_1(k), x_2(k), \cdots, x_N(k)]^{\mathrm{T}},$$
$$\boldsymbol{v}(k) = [v_1(k), v_2(k), \cdots, v_N(k)]^{\mathrm{T}},$$
$$\boldsymbol{u}(k) = [u_1(k), u_2(k), \cdots, u_N(k)]^{\mathrm{T}}.$$

Definition 1 A stationary consensus is achieved if for any initial position and velocity states, and any i, $j \in 1, 2, \cdots, N$, the following equalities hold:

$$\lim_{k\to\infty}\left[x_i(k) - x_j(k)\right] = 0, \quad \lim_{k\to\infty} v_i(k) = 0.$$

Definition 2 A dynamic consensus is achieved if for any initial position and velocity states, and any i, $j \in 1, 2, \cdots, N$, the following equalities hold:

$$\lim_{k\to\infty}\left[x_i(k) - x_j(k)\right] = 0,$$
$$\lim_{k\to\infty}\left[v_i(k) - v_j(k)\right] = 0.$$

3.1 Stationary consensus case

To solve the stationary consensus problem of second-order MAS, a general stationary consensus protocol is used in this paper:

$$u_i(k) = -\kappa_s v_i(k) + \sum_{j=1}^{N} a_{ij}[x_j(k-\tau_k) - x_i(k-\tau_k)] \tag{4}$$

where $\kappa_s \in \mathbb{R}^+$ is the velocity damping gain.

Denoting $\tilde{x}_i(k) = x_i(k) - x_1(k)$ and $\tilde{v}_i(k) = v_i(k) - v_1(k)$, the position and velocity error vectors can be represented as

$$\tilde{\boldsymbol{x}}(k) = [\tilde{x}_2(k), \tilde{x}_3(k), \cdots, \tilde{x}_N(k)]^{\mathrm{T}},$$
$$\tilde{\boldsymbol{v}}(k) = [\tilde{v}_2(k), \tilde{v}_3(k), \cdots, \tilde{v}_N(k)]^{\mathrm{T}}.$$

We can transform Eq. (3) into

$$\begin{bmatrix} \tilde{\boldsymbol{x}}(k+1) \\ \tilde{\boldsymbol{v}}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{I} & (h - \frac{h^2}{2}\kappa_s)\boldsymbol{I} \\ \boldsymbol{0} & (1 - h\kappa_s)\boldsymbol{I} \end{bmatrix}}_{\boldsymbol{A}_{\mathrm{s}}} \begin{bmatrix} \tilde{\boldsymbol{x}}(k) \\ \tilde{\boldsymbol{v}}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} -\frac{h^2}{2}\hat{\boldsymbol{L}} & \boldsymbol{0} \\ -h\hat{\boldsymbol{L}} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{B}_{\mathrm{s}}} \begin{bmatrix} \tilde{\boldsymbol{x}}(k-\tau_k) \\ \tilde{\boldsymbol{v}}(k-\tau_k) \end{bmatrix} \tag{5}$$

where

$$\hat{\boldsymbol{L}} = \begin{bmatrix} l_{22} - l_{12} & l_{23} - l_{13} & \cdots & l_{2N} - l_{1N} \\ l_{32} - l_{12} & l_{33} - l_{13} & \cdots & l_{3N} - l_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ l_{N2} - l_{12} & l_{N3} - l_{13} & \cdots & l_{NN} - l_{1N} \end{bmatrix}.$$

With $\boldsymbol{\xi}(k) = [\tilde{\boldsymbol{x}}^{\mathrm{T}}(k), \tilde{\boldsymbol{v}}^{\mathrm{T}}(k)]^{\mathrm{T}}$, Eq. (5) can be represented as

$$\boldsymbol{\xi}(k+1) = \boldsymbol{A}_{\mathrm{s}}\boldsymbol{\xi}(k) + \boldsymbol{B}_{\mathrm{s}}\boldsymbol{\xi}(k-\tau_k) \tag{6}$$

3.2 Dynamic consensus case

We use the following protocol to solve the dynamic consensus problem for second-order MAS:

$$u_i(k) = \sum_{j=1}^{N} a_{ij}[x_j(k-\tau_k) - x_i(k-\tau_k)] + \kappa_{\mathrm{d}}\sum_{j=1}^{N} a_{ij}[v_j(k-\tau_k) - v_i(k-\tau_k)] \tag{7}$$

where $\kappa_{\mathrm{d}} \in \mathbb{R}^+$ is a positive control gain.

Similar to the stationary consensus case, by transforming Eq. (3) into the error vector form, we obtain

$$\begin{bmatrix} \tilde{\boldsymbol{x}}(k+1) \\ \tilde{\boldsymbol{v}}(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{I} & h\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{A}_{\mathrm{d}}} \begin{bmatrix} \tilde{\boldsymbol{x}}(k) \\ \tilde{\boldsymbol{v}}(k) \end{bmatrix} + \underbrace{\begin{bmatrix} -\dfrac{h^2}{2}\hat{\boldsymbol{L}} & -\dfrac{\kappa_{\mathrm{d}}h^2}{2}\hat{\boldsymbol{L}} \\ -h\hat{\boldsymbol{L}} & -\kappa_{\mathrm{d}}h\hat{\boldsymbol{L}} \end{bmatrix}}_{\boldsymbol{B}_{\mathrm{d}}} \begin{bmatrix} \tilde{\boldsymbol{x}}(k-\tau_k) \\ \tilde{\boldsymbol{v}}(k-\tau_k) \end{bmatrix} \tag{8}$$

Then, we have

$$\boldsymbol{\xi}(k+1) = \boldsymbol{A}_{\mathrm{d}}\boldsymbol{\xi}(k) + \boldsymbol{B}_{\mathrm{d}}\boldsymbol{\xi}(k-\tau_k) \tag{9}$$

Remark 2 In the stationary consensus protocol (Eq. (4)), $-\kappa_{\mathrm{s}}v_i(k)$ acts as a damping component, which guarantees that the velocity of the agent converges to zero. By transforming Eq. (3) into Eqs. (6) and (9), the original system consensusability problem is turned into the stabilization problems of two error systems.

4 Main Results

Unifying Eqs. (6) and (9) gives rise to

$$\boldsymbol{\xi}(k+1) = \boldsymbol{A}\boldsymbol{\xi}(k) + \boldsymbol{B}\boldsymbol{\xi}(k-\tau_k) \tag{10}$$

with $(\boldsymbol{A}, \boldsymbol{B}) \in \{(\boldsymbol{A}_{\mathrm{s}}, \boldsymbol{B}_{\mathrm{s}}), (\boldsymbol{A}_{\mathrm{d}}, \boldsymbol{B}_{\mathrm{d}})\}$ (see Eqs. (5) and (8) for the definitions of $\boldsymbol{A}_{\mathrm{s}}, \boldsymbol{B}_{\mathrm{s}}, \boldsymbol{A}_{\mathrm{d}}$, and $\boldsymbol{B}_{\mathrm{d}}$).

By combing stationary and dynamic consensus problems under the same framework, a unified method can be used for second-order MAS consensusability analysis. We are now in the position to present the sufficient condition for consensus.

Theorem 1 For system Eq. (10) subject to time-varying delays, and under Assumptions 1 and 2, stationary and dynamic consensus can be achieved for time-varying interval-like delays $\tau_k \in [\tau_m, \tau_M]$ if there exist positive matrices $\boldsymbol{P}_i > 0\,(i = 1, 2, 3)$, $\boldsymbol{Q}_j > 0\,(j = 1, 2, 3)$, and matrix $\boldsymbol{R}$ with appropriate dimensions such that the following matrix inequalities hold:

$$\boldsymbol{\Xi} - \boldsymbol{\zeta}^{\mathrm{T}} \boldsymbol{\Lambda} \boldsymbol{\zeta} < 0 \tag{11}$$

$$\boldsymbol{\Lambda} > 0 \tag{12}$$

$$\boldsymbol{\Xi} = (\boldsymbol{A}\boldsymbol{e}_1 + \boldsymbol{B}\boldsymbol{e}_2)^{\mathrm{T}} \boldsymbol{P}_1 (\boldsymbol{A}\boldsymbol{e}_1 + \boldsymbol{B}\boldsymbol{e}_2) + \boldsymbol{e}_1^{\mathrm{T}} [-\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 + (\tau_M - \tau_m + 1)\boldsymbol{Q}_1] \boldsymbol{e}_1 - \boldsymbol{e}_2^{\mathrm{T}}\boldsymbol{Q}_1\boldsymbol{e}_2 - \boldsymbol{e}_3^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{e}_3 - \boldsymbol{e}_4^{\mathrm{T}}\boldsymbol{P}_3\boldsymbol{e}_4 + [(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{e}_1 + \boldsymbol{B}\boldsymbol{e}_2]^{\mathrm{T}} \cdot [\tau_m^2 \boldsymbol{Q}_2 + (\tau_M - \tau_m)^2 \boldsymbol{Q}_3] [(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{e}_1 + \boldsymbol{B}\boldsymbol{e}_2] - (\boldsymbol{e}_1 - \boldsymbol{e}_3)^{\mathrm{T}} \boldsymbol{Q}_2 (\boldsymbol{e}_1 - \boldsymbol{e}_3),$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{Q}_3 & \boldsymbol{R} \\ \boldsymbol{R}^{\mathrm{T}} & \boldsymbol{Q}_3 \end{bmatrix},$$

$$\boldsymbol{\zeta} = [\boldsymbol{e}_3^{\mathrm{T}} - \boldsymbol{e}_2^{\mathrm{T}}, \boldsymbol{e}_2^{\mathrm{T}} - \boldsymbol{e}_4^{\mathrm{T}}],$$

where $\boldsymbol{e}_i, i = 1, 2, 3, 4$ are block entry matrices, e.g., $\boldsymbol{e}_2 = [\boldsymbol{0}, \boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}]$.

Proof For system Eq. (10), we construct the Lyapunov functional candidate as follows:

$$V(k) = \sum_{i=1}^{4} V_i(k),$$

where

$$V_1(k) = \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_1\boldsymbol{x}(k),$$

$$V_2(k) = \sum_{i=k-\tau_m}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{P}_2\boldsymbol{x}(i) + \sum_{i=k-\tau_M}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{P}_3\boldsymbol{x}(i),$$

$$V_3(k) = \sum_{i=k-\tau_k}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) + \sum_{j=\tau_m}^{\tau_M-1} \sum_{i=k-j}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i),$$

$$V_4(k) = \sum_{j=0}^{\tau_m-1} \sum_{i=k-1-j}^{k-1} \tau_m \boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_2\boldsymbol{\delta}(i) + \sum_{j=\tau_m}^{\tau_M-1} \sum_{i=k-1-j}^{k-1} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i).$$

Then we have

$$\Delta V_1(k) = (\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{x}(k-\tau_k))^{\mathrm{T}}\boldsymbol{P}_1(\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{x}(k-\tau_k)) - \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_1\boldsymbol{x}(k) = \boldsymbol{x}^{\mathrm{T}}(k)(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x}(k) + \boldsymbol{x}^{\mathrm{T}}(k-\tau_k)\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{B}\boldsymbol{x}(k-\tau_k) + \boldsymbol{x}^{\mathrm{T}}(k-\tau_k)\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{B}\boldsymbol{x}(k-\tau_k),$$

$$\Delta V_2(k) = \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_2\boldsymbol{x}(k) + \boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{P}_3\boldsymbol{x}(k) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_m)\boldsymbol{P}_2\boldsymbol{x}(k-\tau_m) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_M)\boldsymbol{P}_3\boldsymbol{x}(k-\tau_M),$$

$$\Delta V_3(k) = (\tau_M - \tau_m + 1)\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{Q}_1\boldsymbol{x}(k) - \sum_{i=k+1-\tau_M}^{k-\tau_m} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) + \sum_{i=k+1-\tau_{k+1}}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) - \sum_{i=k-\tau_k}^{k-1} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) \leqslant (\tau_M - \tau_m + 1)\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{Q}_1\boldsymbol{x}(k) - \sum_{i=k+1-\tau_M}^{k-\tau_m} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) + \sum_{i=k+1-\tau_M}^{k-\tau_m} \boldsymbol{x}^{\mathrm{T}}(i)\boldsymbol{Q}_1\boldsymbol{x}(i) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_k)\boldsymbol{Q}_1\boldsymbol{x}(k-\tau_k) = (\tau_M - \tau_m + 1)\boldsymbol{x}^{\mathrm{T}}(k)\boldsymbol{Q}_1\boldsymbol{x}(k) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_k)\boldsymbol{Q}_1\boldsymbol{x}(k-\tau_k),$$

$$\Delta V_4(k) = \boldsymbol{\delta}^{\mathrm{T}}(k)[\tau_m^2\boldsymbol{Q}_2 + (\tau_M - \tau_m)^2\boldsymbol{Q}_3]\boldsymbol{\delta}(k) - \sum_{i=k-\tau_m}^{k-1} \tau_m\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_2\boldsymbol{\delta}(i) - \sum_{i=k-\tau_M}^{k-\tau_m-1} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i) \tag{13}$$

By Lemma 1, we can obtain

$$-\left(\sum_{i=k-\tau_m}^{k-1} \boldsymbol{\delta}^{\mathrm{T}}(i)\right)\boldsymbol{Q}_2\left(\sum_{i=k-\tau_m}^{k-1} \boldsymbol{\delta}(i)\right) = -(\boldsymbol{x}(k) - \boldsymbol{x}(k-\tau_m))^{\mathrm{T}}\boldsymbol{Q}_2(\boldsymbol{x}(k) - \boldsymbol{x}(k-\tau_m)) \geqslant -\sum_{i=k-\tau_m}^{k-1} \tau_m\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_2\boldsymbol{\delta}(i) \tag{14}$$

Similarly, we have

$$-\sum_{i=k-\tau_M}^{k-\tau_m-1} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i) = -\sum_{i=k-\tau_M}^{k-1-\tau_k} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i) - \sum_{i=k-\tau_k}^{k-\tau_m-1} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i) \leqslant -\beta_1^{-1}(\boldsymbol{x}(k-\tau_m) - \boldsymbol{x}(k-\tau_k))^{\mathrm{T}}\boldsymbol{Q}_3(\boldsymbol{x}(k-\tau_m) - \boldsymbol{x}(k-\tau_k)) - \beta_2^{-1}(\boldsymbol{x}(k-\tau_k) - \boldsymbol{x}(k-\tau_M))^{\mathrm{T}}\boldsymbol{Q}_3(\boldsymbol{x}(k-\tau_k) - \boldsymbol{x}(k-\tau_M)),$$

where $\beta_1 = \dfrac{\tau_k - \tau_m}{\tau_M - \tau_m}$, $\beta_2 = \dfrac{\tau_M - \tau_k}{\tau_M - \tau_m} = 1 - \beta_1$.

If there exists $\boldsymbol{R}$ such that Eq. (12) holds, the following inequality can be obtained by Lemma 2:

$$-\sum_{i=k-\tau_M}^{k-\tau_m-1} (\tau_M - \tau_m)\boldsymbol{\delta}^{\mathrm{T}}(i)\boldsymbol{Q}_3\boldsymbol{\delta}(i) \leqslant -\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{\gamma} \tag{15}$$

where

$$\gamma = \left[\boldsymbol{x}^{\mathrm{T}}(k-\tau_m) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_k), \boldsymbol{x}^{\mathrm{T}}(k-\tau_k) - \boldsymbol{x}^{\mathrm{T}}(k-\tau_M)\right]^{\mathrm{T}}.$$

It should be noted that when $\tau_k = \tau_m$ or $\tau_k = \tau_M$, Formula (15) still holds. With Formulas (14) and (15), we have

$$\Delta V_4(k) \leqslant \boldsymbol{\delta}^{\mathrm{T}}(k)[\tau_m^2 \boldsymbol{Q}_2 + (\tau_M - \tau_m)^2 \boldsymbol{Q}_3]\boldsymbol{\delta}(k) - (\boldsymbol{x}(k) - \boldsymbol{x}(k-\tau_m))^{\mathrm{T}} \boldsymbol{Q}_2 (\boldsymbol{x}(k) - \boldsymbol{x}(k-\tau_m)) - \boldsymbol{\zeta}^{\mathrm{T}} \boldsymbol{\Lambda} \boldsymbol{\zeta}.$$

Denoting an augmented vector as $\boldsymbol{\xi}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \boldsymbol{x}^{\mathrm{T}}(k-\tau_k), \boldsymbol{x}^{\mathrm{T}}(k-\tau_m), \boldsymbol{x}^{\mathrm{T}}(k-\tau_M)]^{\mathrm{T}}$, we can obtain that

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \leqslant \boldsymbol{\xi}^{\mathrm{T}}(k)\{\boldsymbol{\Xi} - \boldsymbol{\zeta}^{\mathrm{T}} \boldsymbol{\Lambda} \boldsymbol{\zeta}\}\boldsymbol{\xi}(k).$$

If Formula (11) holds, $\Delta V(k) < 0$. Thus the system Eq. (10) is asymptotically stable, which means stationary and dynamic consensus are achieved by the second-order MAS with time-varying interval-like delays $\tau_k \in [\tau_m, \tau_M]$ using protocols Eqs. (4) and (7), respectively. □

Remark 3 Since the communication graph contains antagonistic interactions modeled by negative weights, the properties of SIA matrices can no longer be explored to solve the time delay problem addressed in this paper. The construction of the Lyapunov functional is in light of Ref. [16]. Theorem 1 is derived with the reciprocally convex approach[15], which is widely used in reducing the matrix variables to alleviate the computational burden of the LMI criterion.

5 Illustrative Examples

In this section, illustrative examples are presented to verify the proposed result. Here we consider a second-order MAS with four agents. Without loss of generality, the interactions of agent 4 towards agent 1 and agent 3 towards agent 2 are assumed to be antagonistic, which are represented by dashed lines on the communication network topology (see Fig. 1). The time-delay is lower bounded by 10 and randomly switching according to uniform distribution. The adjacent matrix of the system topology is set as follows:

$$\mathcal{A} = \begin{bmatrix} 0 & 0.8 & 0 & -0.2 \\ 0.5 & 0 & -0.5 & 0 \\ 0.3 & 0 & 0 & 0.6 \\ 0.8 & 0.2 & 0 & 0 \end{bmatrix}.$$

Initial states of the five agents are set to be:

$$\boldsymbol{x}(0) = [\ 1 \quad 3 \quad 0 \quad 2\ ]^{\mathrm{T}},$$
$$\boldsymbol{v}(0) = [\ 0.2 \quad -0.2 \quad 0 \quad 0.1\ ]^{\mathrm{T}}.$$

For the stationary consensus case with $\kappa_{\mathrm{s}} = 2$, the upper bound of the delay satisfying the consensus criterion of Theorem 1 is 153. Figure 2 shows the position and velocity trajectories of all agents with time-varying delays within [10, 153]. The agents' positions asymptotically converge to the same value, and the velocities converge to zero. Thus, a stationary consensus is achieved.

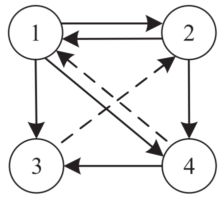


Fig. 1 System topology.

For the dynamic consensus case, the maximum allowable delay is 39 when $\kappa_{\mathrm{d}} = 2$. Figure 3 shows the position and velocity trajectories of all agents subject to time-varying delays and antagonistic interactions. It is readily seen that a dynamic consensus is achieved.

6 Conclusions

The consensus problem of second-order MAS subject to time-varying interval-like input delays is addressed in this paper. The notion of consensus is extended to networks containing antagonistic interactions modeled by negative weights on the communication graph. Both

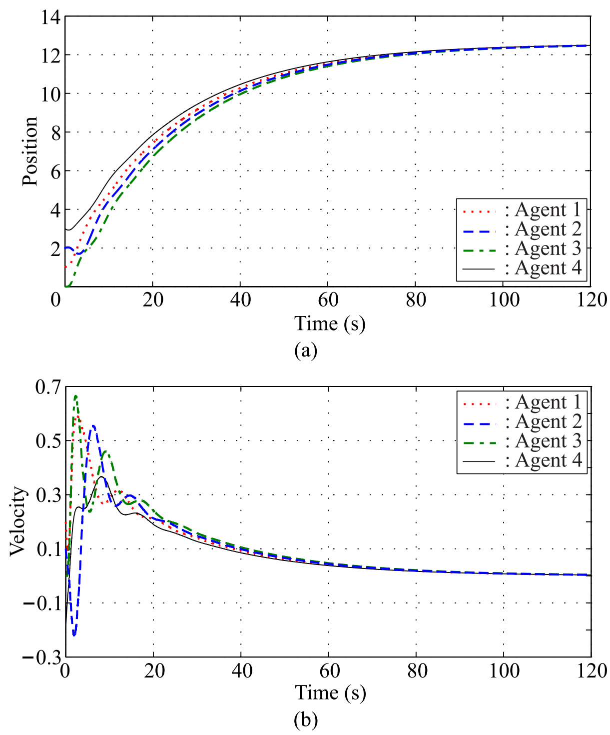
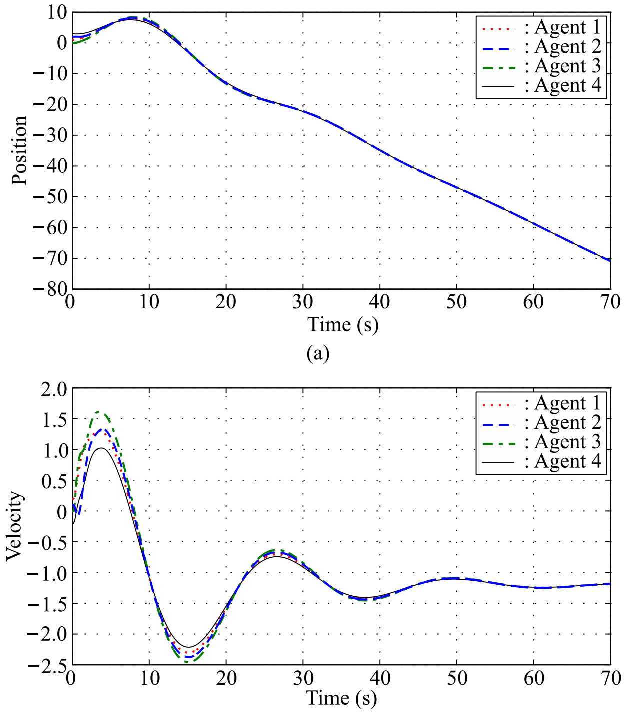


Fig. 2 Stationary consensus of the agents with time-varying delays.



(a)

(b)

Fig. 3 Dynamic consensus of the agents with time-varying delays.

stationary and dynamic consensus cases have been considered under a unified framework. A criterion for consensus has been derived in terms of LMIs. At the end of the paper, the effectiveness of the proposed approach has been verified via examples.

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